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Inevitable self-similar topology of binary trees and their diverse hierarchical density

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Abstract. Self-similar topology, which can be characterized as power law size distribution, has been found in diverse tree networks ranging from river networks to taxonomic trees. In this study, we find that the statistical self-similar topology is an inevitable consequence of any full binary tree organization. We show this by coding a binary tree as a unique bifurcation string. This coding scheme allows us to investigate trees over the realm from deterministic to entirely random trees. To obtain partial random trees, partial random perturbation is added to the deterministic trees by an operator similar to that used in genetic algorithms. Our analysis shows that the hierarchical density of binary trees is more diverse than has been described in earlier studies. We find that the connectivity structure of river networks is far from strict self-similar trees. On the other hand, organization of some social networks is close to deterministic supercritical trees.

PACS. 89.75.Da Systems obeying scaling laws – 89.75.Hc Networks and genealogical trees – 89.75.Fb Structures and organization in complex systems – 05.45.Df Fractals

1 Introduction

The notion of self-similarity has significantly impacted network studies over the broad range of disciplines such as biology, chemistry, earth science, economics, engineering, hydrology, physics, and even sociology. Self-similar features are found in most binary tree networks, that develop in open dissipative systems [1], such as river networks [2], blood vessels [3,4], vascular organizations in plants [5,6], agglomerates of charged metal particles in castor oil [7], and even lightning [8], and have attracted significant attention. The self-similar topology of binary tree networks also serves as a motif of the organization of general complex networks such as social networks. This is because most complex networks have community structure, and they can be transformed into equivalent binary tree networks as a result of the community organization [9, 10]. Recent studies reported that these transformed binary trees also exhibit self-similarity [11,12].

Here, we may ask: why is statistical self-similar topology found in such diverse binary tree networks? A corollary to this question is: is there a unique connectivity structure that gives rise to the statistical self-similarity? If that is the case, then there should also be a comparable number of the other types of tree networks that are clearly distinguished from trees of self-similar topology. In this paper, we investigate this issue which leads to the hypothesis that statistical self-similar topology is an inevitable characteristic that arises from almost any connectivity structure of tree networks.

As a measure of self-similarity, we use the power law tendency of (exceedance) size distributions [13–16], i.e.,

$$P(\delta) \propto \delta^{-\epsilon-1} \quad \text{and} \quad P(\varDelta \geq \delta) \propto \delta^{-\epsilon} \qquad (1)$$

identified as the probability and the exceedance probability distribution, respectively, of the sub-tree size δ, over an entire tree network. Interesting power law (exceedance) size distributions have been found in river networks with fairly constant exponents ($\epsilon = 0.43 \pm 0.03$) [14]. Various social networks, when cast into equivalent binary tree networks using community organization, also exhibit power law in their (exceedance) size distributions [11,12]. However, noticeable difference was found in their exponents ϵ. Guimerà et al. [11] identified community structure of an e-mail network of the University at Rovira i Virgili and found that the transformed binary tree e-mail network shows the power law exceedance size distribution with exponent $\epsilon = 0.48$, a value interestingly close to that of river networks. A power law size distribution with a similar exponent is also found in a transformed network of Jazz musicians [12]. However, binary transformed networks of

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scientists exhibit power law exceedance size distributions with exponents $\epsilon \approx 1$ [12], very different from that of river networks.

These analyses of real networks motivate us to investigate not only the power law itself but also the specific values of exponents ϵ. The exponent ϵ indicates the degree of the density in hierarchical structure. Higher density characterizes trees with higher number of edges closer to the root node. The denser the hierarchy of a tree is (such as strict self-similar tree), the steeper the (exceedance) size distribution on a log-log scale, in turn resulting in greater ϵ. The organization of binary trees that enables these interesting observations in their (exceedance) size distributions, i.e., ubiquitous power law tendency with distinct variation in exponents, remains unknown [11,12].

Recently, the density of hierarchical structure has been actively studied but consistent conclusions are not made yet. Based on the fitted exponents of observed binary trees listed above, Arenas et al. [12] postulated the existence of two distinct classes of binary trees based on their size distribution, i.e., one with $\epsilon \approx 0.5$ and the other with $\epsilon \approx 1$. However, this empirical speculation is based on a limited number of binary trees, therefore requiring further validation. On the other hand, Caldarelli et al. [17] argued that any treelike representation should lead to $\epsilon \approx 1$. They admitted the occurrence of $\epsilon \approx 0.5$ found in some networks listed above as exceptions but stressed that ϵ other than 1 and 0.5 cannot be found. However, their study is limited to a specific class of trees, i.e., completely random trees. It is important to note that studies on complex networks show that most networks are subject to the organization, which is neither purely ordered nor completely random but somewhere in between [18]. The connectivity structure of these partially random tree networks, despite their prevalence, has not been investigated.

In this paper, we study the connectivity structure of various binary tree networks, ranging from deterministic to entirely random, by investigating their power law tendency in (exceedance) size distributions and the variation of the exponents. As a way to represent the structure of binary tree networks, we use a 'bifurcation string' approach which is analogous to former binary string approaches [19–21]. The bifurcation string is composed of 'bifurcation indexes', either 1 or 0, indicating if an edge in a binary tree bifurcates or not as one moves from the root node upstream. Similar to the DNA string in the chromosome, a unique bifurcation string contains sufficient information to reproduce the topology of a given binary tree.

We investigate three types of bifurcation strings: deterministic strings, entirely random strings, and strings intermediate between these two extremes. Deterministic strings have strict regularity in the sequence of bifurcation indexes. On the other hand, entirely random strings have randomly generated bifurcation indexes. Since observed networks are not limited to these two types, we also investigate the realm between these extremes. Strings in this intermediate realm are generated by an operator similar to that used in genetic algorithms. This enables us to draw a map, which locates various binary trees with the corresponding characteristics (exceedance size distributions in this paper), in a wide range of randomness from the deterministic (where randomness is zero) to the entirely random organizations. This provides a macroscopic view since specific networks that have been the focus of earlier studies are merely points in this map.

The rest of this paper is organized into four sections. In Section 2, we clarify terminologies used in this study since different (and often confusing) terminologies are used for network studies in various disciplines. In Section 3, we introduce the bifurcation string methodology devised for analyzing binary tree networks and show its applicability for ideal trees, i.e., entirely deterministic and entirely random trees. The (exceedance) size distribution of binary trees that partially contain random connectivity is discussed in Section 4. The conclusions are given in Section 5.

2 Binary trees and size distributions

2.1 Network terminologies

Networks are composed of nodes, and edges that connect the nodes. Defining a consecutive set of edges as a path, a loop (or a cycle) is formed in the network if there is more than one path between any two nodes. Based on the existence of loops, networks can be divided into two types, i.e., networks with and without loops. A loopless network is also called a tree network or simply a tree.

In a tree, nodes are classified into three types: end-nodes (leaf or terminal nodes), the root, and internal nodes. The end-node has only one attached edge. The root is an initially existing node from where the network starts to expand. The other nodes are called internal nodes. For convenience, we define the upstream and the downstream as the directions toward the end-nodes and the root, respectively. Nodes located at the upstream of a node are its sub-nodes. The root has the greatest number of sub-nodes in the network. Sub-nodes that are directly connected to a node are called children (or offspring) of the node, and the node is called their parent. Only a single edge exists between a parent and its each child. The number of edges between a node and the root along the path between these two is the "generation" of the node.

End-nodes have zero children while the other nodes have k children ($k > 0$). If k is at best two, the network is often called the binary tree. A tree where k is constant is called the 'full tree' and a tree with k fixed as two is called the 'full binary tree' (Fig. 1). In this paper, we discuss only the full binary tree. Various networks that develop in open dissipative systems (examples listed in introduction) are characterized as full binary trees if only junctions are considered as internal nodes, which is the concept used in the Horton-Strahler ordering scheme [22]. The 'average' number of children per node in a tree is termed as the "branching ratio" B_r [23]. Based on their branching ratio, trees are classified [23] as subcritical ($B_r < 1$), critical ($B_r = 1$), and supercritical ($B_r > 1$) trees.

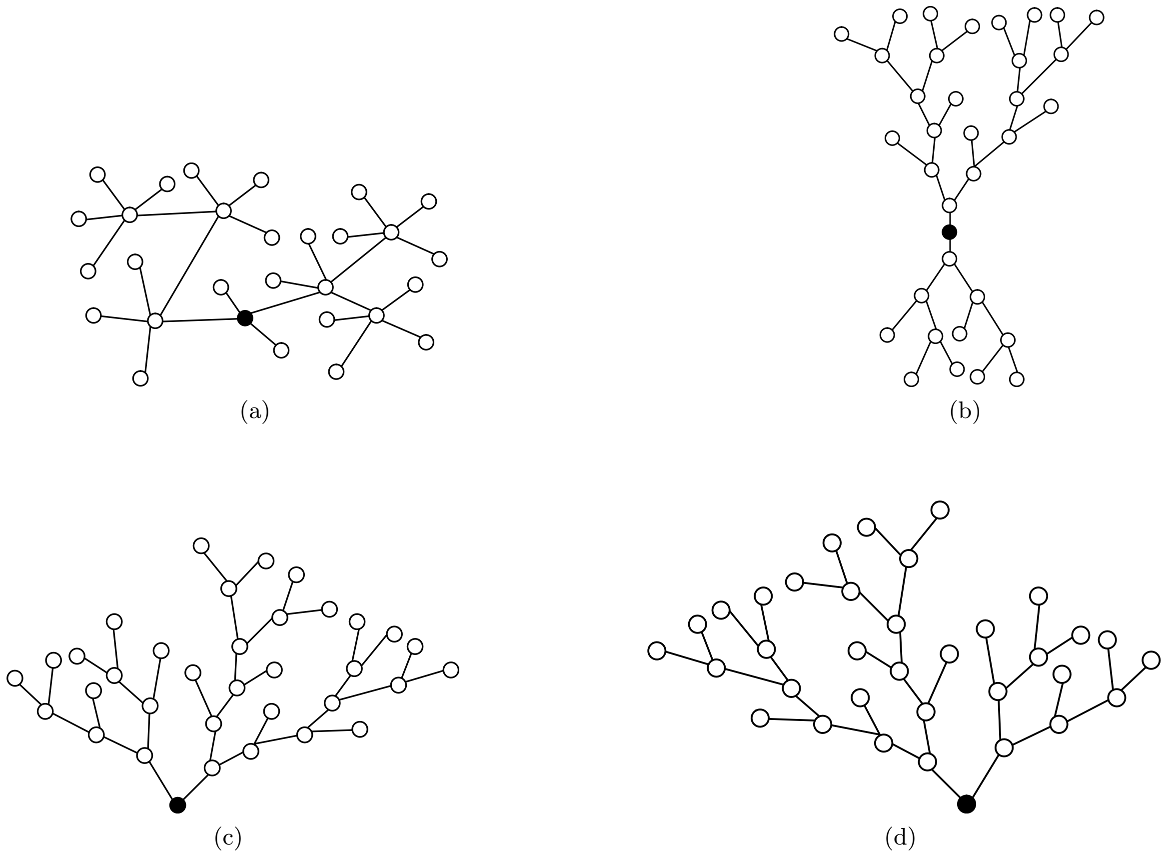

Fig. 1. Examples of tree (loopless) networks. Black nodes are roots and white nodes are end-nodes or internal nodes. (a) A tree network in which the number of children k for the root and internal nodes is fixed as four. (b) A tree network in which k is fixed as two. This network is termed as a full binary tree. Stem-root structure of plants (in 2-D illustration) follows this topology. (c) A modified binary tree shown in (b). Note that this is topologically equivalent to (b) following Shreve's [28] definition. River networks without delta formation follow this topology. (d) A mirror-imaged binary tree of (c). This is topologically distinct from (b) and (c) following Shreve's [28] definition.

2.2 Size distribution

The probability distributions of the sub-tree size, where a sub-tree is defined as a portion of the entire tree with any node being a root, have been used in characterizing the topology of various tree networks. The 'size' δ refers to either 'load' δ_l or 'magnitude' δ_m which are defined as follows. Let us consider the flux of energy, information, or matter over a tree network. Each node has its own 'input' (the quantity of energy, information, or matter), which then flows downstream. The input can be the amount of data packet in internet or the rainfall excess in river networks. The definition of input is limited to those given from outside of the network and excludes those transferred from other nodes.

The load δ_l of a node [24] is defined as the total amount of inputs flowing out of the node. Therefore, δ_l is the sum of inputs of all sub-nodes as well as that of the node itself. The load of an end-node is the same as its input. The concept of load is useful in analysis of networks whose nodes have physical 'inputs', such as river networks. River networks are spanning trees and exactly embeddable in 2-D lattice where the input is the rainfall excess and the load of a node can be the amount of streamflow (precisely, direct runoff) at the location. The load distribution $P(\delta_l)$ is defined as the probability distribution that a node has a certain load δ_l. Then, we can obtain its exceedance probability $P(\Delta \geq \delta_l)$.

The magnitude of a node δ_m is the total amount of inputs that come from only end-nodes among the node and its sub-nodes. This terminology is similar to that of Shreve [19] for stream links. If the node itself is an end-node, its magnitude is the same as its input. For nodes other than the end-nodes, δ_m is the sum of inputs of end-nodes upstream of the node. The difference between the load and the magnitude is illustrated in Figure 2.

Topological organization of general complex networks with loops can be simplified as binary trees by hierarchical grouping of densely connected nodes (the procedure called community identification) [9,10]. The concept of the magnitude is used in analyzing such binary trees [11,12] in which only end-nodes represent the nodes of the original networks while all internal nodes and the root are dummy

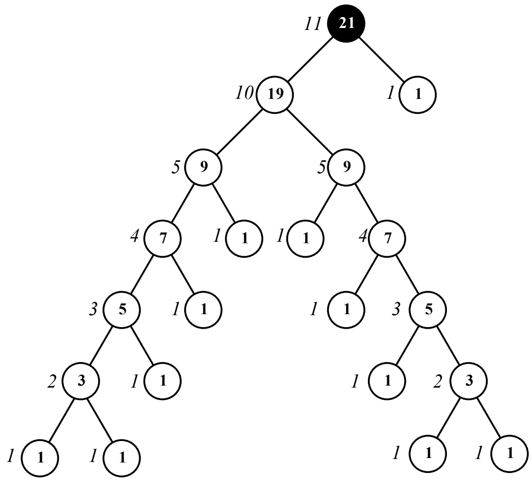

Fig. 2. Illustration to show the difference between the load and the magnitude. In this example, the input for every node is unity. The load (displayed inside each node) and the magnitude (printed as italic at the side of each node) are counted for each node in an example full binary tree.

nodes. Similar to the definition of $P(\delta_l)$ and $P(\Delta \geq \delta_l)$, we can define the magnitude distribution $P(\delta_m)$ as the probability distribution that a node has a certain magnitude δ_m and its exceedance probability distribution $P(\Delta \geq \delta_m)$.

Above definitions for the load δ_l and the magnitude δ_m are general. Narrower definitions, which do not require the concept of the flux and the input, can be also used: the load of a node is simply the number of all subnodes as well as the node itself, and the magnitude is the number of end-nodes upstream unless the node is an end-node where $\delta_m = 1$. These definitions are equivalent to the general definitions under the case that the input for every node is unity. This condition of spatially uniform input is often assumed for river networks, where direct runoff (the load) has been estimated from the contributing area under the assumption of spatially constant rainfall excess (the input). Since both the load and the magnitude simply represent the size of a sub-tree under the narrower definitions, we may call these as the size in a general context (the way Guimerà et al. [11] used). Similarly, $P(\delta_l)$ and $P(\delta_m)$ are simply called as size distributions and $P(\Delta \geq \delta_l)$ and $P(\Delta \geq \delta_m)$ are called exceedance size distributions. Size and exceedance size distributions can be obtained by counting the number of nodes in the network having δ_l load ($N(\delta_l)$) or δ_m magnitude ($N(\delta_m)$). In this study, we follow this definition of size, i.e., the input for every node is unity.

Note that, except for end-nodes, the load is greater than the magnitude (Fig. 2). For example, if the input for every node is unity, the maximum loads M_l (δ_l of the root) of trees in Figures 1a and 1b are 29 and 35, respectively. On the other hand, the maximum magnitudes M_m (δ_m of the root) of the same trees are 22 and 18, respectively. Nevertheless, this difference has little effect on the exponents (ϵ) of power law (exceedance) size distributions. For a simple case that the input for every node is unity, we can derive $\delta_l = 2\delta_m - 1$ for full binary trees. If a load distribution follows $P(\delta_l) \propto {\delta_l}^{-\epsilon-1}$, then $P(\delta_m) \propto (2\delta_m - 1)^{-\epsilon-1}$. Inverse of this relationship also holds, i.e., if $P(\delta_m) \propto {\delta_m}^{-\epsilon-1}$, then $P(\delta_l) \propto (\delta_l + 1)^{-\epsilon-1}$. Therefore, for δ_l (or δ_m) $\gg 1$, if one of the load and the magnitude distributions follows a power law, the other distribution also follows the power law with the same exponent value. Similarly, if either $P(\Delta \geq \delta_l)$ or $P(\Delta \geq \delta_m)$ follows a power function, the other also follows the power law with the same exponent ϵ. This property yields practical convenience: we need to analyze only one of either the load or the magnitude. In this study, we only analyze the load of theoretical trees and refer to this as the size.

Establishing an analytical derivation of power law size distributions, observed in such diverse trees, has been a challenging task. A popular approach for theoretical analysis of binary tree topology is the systematic ordering of binary trees based on the Horton-Strahler ordering scheme [22]. This scheme is originally devised for river networks but can be applied to any full binary tree [11,12]. Based on this ordering, the average load $\overline{\delta}_\omega$ and the average number $\overline{N}_\omega$ of branches of order ω can be defined. Well-known characteristics of these quantities observed in river networks are Horton's laws. They state nicely fitted log-linear relationships of $\overline{\delta}_\omega$ and $\overline{N}_\omega$ as functions of the order ω, i.e., $\overline{\delta}_\omega / \overline{\delta}_{\omega-1} \approx R_A$ and $\overline{N}_\omega / \overline{N}_{\omega+1} \approx R_B$ where constants R_A and R_B are Horton ratios [25,26].

For trees where Horton's laws hold, we can show that the load distributions follow exact power functions, and exceedance load distributions are asymptotically power functions (see Appendix A). As discussed above, these can be generalized as power law (exceedance) size distributions. In fact, it was shown that almost all randomly generated full binary trees are subject to follow Horton's laws [27]. By combining these, we arrive at a conclusion that the power law tendency in the (exceedance) size distribution is the inevitable result for almost any randomly generated full binary tree.

However, this leaves two fundamental questions. First, we need to know whether above argument shown for entirely random trees can be generalized to other types of trees, i.e., purely deterministic trees and partially random trees. Second, organization of binary trees that yields various values of exponents fitted to the (exceedance) size distribution is not clear yet. In the next sections, we will show that answers to these two key questions are directly related to each other. Specifically, we will use an algorithmic approach to further confirm the inevitability of self-similar topology and to investigate the connectivity structure that yields variation in fitted exponents ϵ.

At this point, it is important to understand the limits of both the (exceedance) size distribution and Horton ratios in distinguishing trees of distinct topology. According to Shreve [28], a network rotated in the plane perpendicular to the projected plane either entirely or partially, is topologically distinct from the original network. However, the (exceedance) size distribution and Horton ratios can be the same for topologically distinct networks. Figures 1c and 1d show examples of trees that are topologically distinct but have the same (exceedance) size distribution and Horton ratios.

3 Bifurcation strings

To generalize the insight gained from the analytical study in the previous section and to investigate the connectivity structure that yields variation in fitted exponents of (exceedance) size distributions, we devise a simple algorithm to generate and analyze full binary trees. The scope of the proposed algorithm is the full tree, i.e., the number of children for the root and internal nodes is fixed as a constant k (examples shown in Fig. 1).

First, we define the 'node index' as the unique identification of each node. The index begins with one for the root and is sequentially numbered according to the closeness to the root (e.g., Fig. 3). For nodes of the same generation, i.e., nodes located the same number of edges apart from the root, the sequence proceeds from left to right nodes under our convention that the root is oriented to the top of the drawing. This satisfies Shreve's [28] definition of topologically distinct channel networks which requires that node indexes for nodes of the same generation should be unique and should not be interchangeable with other nodes of the same generation. However, the resulting size distribution is invariant whether the sequence proceeds from left to right nodes or vice versa.

Then, we pay attention to the fact that all nodes in the full tree fall into one of two types depending on whether they have sub-nodes. End-nodes are the nodes that do not have sub-nodes. However, the root and internal nodes have sub-nodes. This classification provides a basis for representing the topology of full trees in a simple way. We identify these two types by giving the 'bifurcation index' to each node. The bifurcation index is either 0 (to terminate) or 1 (to bifurcate). If the bifurcation index is 0, the node becomes an end-node. On the other hand, if the bifurcation index is 1, k edges, with a sub-node attached to each, are stretched from the node. For a full tree to exist, the root must have the bifurcation index 1. By definition, internal nodes also have the bifurcation index 1.

We can list bifurcation indexes starting from the root in the order of the node index, resulting in a string called the 'bifurcation string'. The bifurcation string is similar to the DNA string in the chromosome, containing all necessary information to reproduce the topology of a full tree. For example, $k = 4$ and the bifurcation string 1;1010;10001100;000100000000;0000 represents the tree network in Figure 1a, $k = 2$ and the bifurcation string 1;11;1111;10100101;01010000;1111;00000000 represents the tree networks in Figures 1b and 1c, and $k = 2$ and the bifurcation string 1;11;1111;01011010;10100000;1111;00000000 represents the tree network in Figure 1d. Here ';'s are inserted in the strings for convenience to distinguish different generations, but serve no other purpose since the knowledge of k is sufficient to identify the generation to which a particular digit belongs. Note that the number of digits of a generation is exactly k times of the number of 1's in the previous generation. For full binary trees, if there are two 1's in a generation, the next generation is composed of four digits.

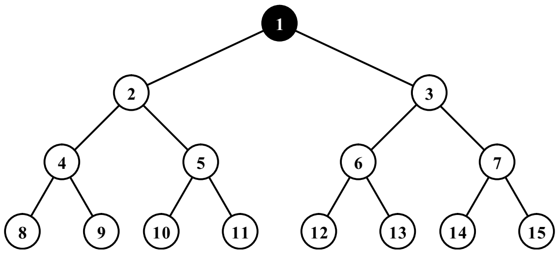

Fig. 3. An example of the strict self-similar binary tree being represented as a completely bifurcating string 1;11;1111;00000000 which repeats a unit sub-string '1' and $n_1 = 2^3 - 1$. The node index is displayed for each node. In this example, $M_l = 15$, $M_m = 8$, $N(\delta_l = 1) = N(\delta_m = 1) = 8$, $N(\delta_l = 3) = N(\delta_m = 2) = 4$, $N(\delta_l = 7) = N(\delta_m = 4) = 2$, and $N(\delta_l = 15) = N(\delta_m = 8) = 1$.

Similar binary string characterizations of full tree networks have been proposed [19–21]. However, the sequence of binary indexes in their binary strings follows a specific traverse direction around every path (called Lukasiewicz's convention). On the other hand, the proposed bifurcation string is sequenced along the node index dependent on the generation. In the following, we will show that this difference provides a new viewpoint in analyzing the full tree networks.

It is simply impossible to analyze the size of an infinitely expanding tree. Defining the number of 1's in a bifurcation string as n_1, for the full tree to be in a finite size, the number of 0's (n_0) in the string must be:

$$n_0 = (k-1)n_1 + 1, \tag{2}$$

which is called the finite size constraint (FSC) in this study. In the case of full binary tree ($k = 2$), this constraint means that $n_0 = n_1 + 1$.

In the following analyses for theoretical trees, we use the narrower definition of size, i.e., the load δ_l is the number of all sub-nodes as well as the node itself, and the magnitude δ_m is the number of end-nodes upstream unless the node is an end-node where $\delta_m = 1$. Without computing the load and the magnitude for every node in a tree, the maximum load M_l and magnitude M_m of a tree can be directly obtained from the bifurcation string. Simply, M_l is the number of total digits ($= n_0 + n_1$) and M_m is the number of 0's ($= n_0$) in a bifurcation string.

Based on the sequence of bifurcation indexes, three types of bifurcation strings exist: deterministic strings, entirely random strings, and strings intermediate of these two extremes. We first investigate topologies corresponding to deterministic and entirely random strings. The intermediate realm will be covered in Section 4.

3.1 Deterministic strings

Deterministic strings have strict regularity in their sequence of bifurcation indexes. One of the most evident regularity is the repetition. If a bifurcation string is composed of the repetitive 'unit sub-string', we call this the 'repetitive string', and the number of digits in the unit sub-string is defined as the 'period'. For example, 1;01;01;01;01;00 is a repetitive string with a unit sub-string '01' of period two. The 1 at the first digit corresponds to the root and the 0's at the last generation are to meet the FSC. These are not considered in the repetition. The number of 0's and 1's in the unit sub-string are termed as n_{s0} and n_{s1}, respectively. Note that the sum of n_{s0} and n_{s1} is the period and the condition $n_{s0} \leq n_{s1}$ is necessary for trees to grow for more than two generations. Therefore, repetitive strings can be grouped into two types: 1. strings with $n_{s0} < n_{s1}$ and 2. strings with $n_{s0} = n_{s1}$. Each of these types is discussed below.

Repetitive strings with $n_{s0} < n_{s1}$, such as 1101101 101101000000 (with unit sub-strings '101'), have more than two 0's attached at the end of the string to meet the FSC. The simplest of these kinds are strings composed of the unit sub-string '1' followed by $n_1 + 1$ digits of 0's (FSC), e.g., 1;11;0000, which are called 'completely bifurcating strings'. The topology of the resulting tree is the "complete binary tree" [29]. Especially, if n_1 can be expressed as $2^i - 1$ where i is a positive integer, the resulting tree is the "perfect binary tree" [30] exhibiting the strict self-similarity or scale-free property (Fig. 3). For perfect binary trees, it is straightforward to show that their (exceedance) size distributions follow power law as:

$$P(\delta) \propto \delta^{-1} \quad \text{and} \quad P(\Delta \geq \delta) \propto \delta^{-1}. \tag{3}$$

Recall that the exponents of power law size distribution and its exceedance distribution differ by unity in equation (1). Perfect binary trees are noticeable exceptions from this rule by showing the same exponent for both distributions (Eq. (3)). Complete binary trees, other than the perfect binary trees, exhibit serrated pattern in their $P(\delta_l)$ and $P(\delta_m)$ distributions. However, their exceedance size distributions are very close to those of the perfect binary trees.

Repetitive strings with $n_{s0} = n_{s1}$ have periods of even numbers and exhibit topology very different from the topology of the previous case ($n_{s0} < n_{s1}$). Examples of unit sub-strings that compose these kinds of bifurcation strings are as follows: '01' and '10' for the period of two; '1100', '0110', and '1001' for the period of four, etc. Since $n_{s0} = n_{s1}$, only two 0's at the last generation satisfy the FSC regardless of the length of the strings. These strings such as 1;10;01;10;01;10;01;00 (with unit sub-string '1001') and 1;11;1000;11;1000;11;1000;00 (with unit sub-string '111000') repeatedly add 'unit sub-trees' to the original trees. The resulting tree looks like a backbone and this topology can be called 'self-repetitive', far from self-similarity (Fig. 4). A repetition of a unit sub-string of $n_{s0} = n_{s1}$ increases δ_l and δ_m of all nodes along the longest path in the original tree by a constant value.

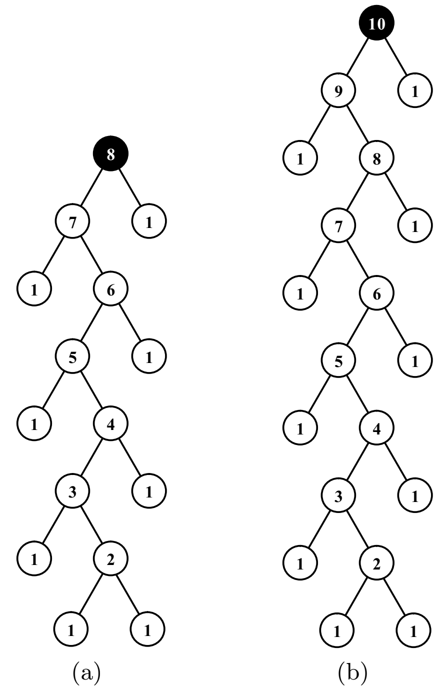

Fig. 4. Example binary trees generated by repetitive strings with a unit sub-string '1001' ($n_{s0} = n_{s1} = 2$, period=4). The magnitude δ_m is displayed for each node. (a) A tree generated by a bifurcation string 1;10;01;10;01;10;01;00. (b) A tree generated by a string 1;10;01;10;01;10;01;10;01;00 which is the string of (a) with one more unit sub-string '1001' added. Addition of a unit sub-string increases M_m from 8 to 10. We compute the number of nodes having δ_m magnitude $N(\delta_m)$ for (a) and (b). Except $N(\delta_m = 1)$, $N(\delta_m)$'s are constant over different δ_m, i.e., $N(\delta_m) = 1$ for $2 \leq \delta_m \leq 8$ in (a) and for $2 \leq \delta_m \leq 10$ in (b). Consequently, for self-repetitive trees generated by these strings, $P(\delta_m)$ is constant, except $P(\delta_m = 1)$, and its exceedance distribution $P(\Delta \geq \delta_m)$ is linear. Comparison between (a) and (b) shows that this is valid regardless of the length of the string. This is because a repetition of a unit sub-string of $n_{s0} = n_{s1}$ increases δ_m of all existing nodes along the longest path by a constant value (in this illustration 2). Therefore, repetitive strings with $n_{s0} = n_{s1}$, regardless of their unit sub-string and length, are subject to follow constant $P(\delta_m)$ and linear $P(\Delta \geq \delta_m)$. Similarly, the same can be proven for their (exceedance) load distribution (constant $P(\delta_l)$ and linear $P(\Delta \geq \delta_l)$).

This means the uniform increase of size over most internal nodes and the root, which results in constant $N(\delta_l)$ and $N(\delta_m)$ for different δ_l and δ_m except a few very small δ_l and δ_m. This indicates constant $P(\delta_l)$ and $P(\delta_m)$ and linear $P(\Delta \geq \delta_l)$ and $P(\Delta \geq \delta_m)$, far from the power law.

It is interesting to note that a completely bifurcating string excluding '1' and '00' of the first and the last digits, respectively (e.g., '1100' in the string '1;11;0000') can be regarded as a unit sub-string of repetitive strings with $n_{s0} = n_{s1}$. Therefore, the size distributions of unit sub-strings with $n_{s0} = n_{s1}$ are close to those of completely bifurcating strings, i.e., the power law. This provides the reason that $P(\Delta \geq \delta_l)$ and $P(\Delta \geq \delta_m)$ of self-repetitive trees deviate from linear trend for very small (of the order of the period of the string) δ_l and δ_m.

Above classification of repetitive strings based on n_{s0} and n_{s1} is interestingly related to the criticality of tree organization. For full trees generated by repetitive strings, the branching ratio is computed as $B_r = kn_{s1}/(n_{s0}+n_{s1})$. Recall that $k = 2$ for full binary trees but the B_r of full binary trees is not necessarily 2 since B_r is the average value including end-nodes which have no children. However, the calculation of B_r excludes the end-nodes at the last generation which are artifact due to the FSC. Strings with $n_{s0} = n_{s1}$ exhibit $B_r = 1$, i.e., critical trees. For example, in trees of Figure 4, every alternate node has two children and the others have no children, resulting in a single child on the average. Therefore, self-repetitive trees also can be called as 'deterministic critical trees'. Completely bifurcating strings where $n_{s0} = 0$ and $n_{s1} = 1$ have $B_r = 2$, i.e., each node has two children unless the growth is interrupted to meet the FSC. Other repetitive strings with $n_{s0} < n_{s1}$ have B_r between 1 and 2. Since repetitive strings with $n_{s0} < n_{s1}$ have B_r always greater than 1, we call the resulting trees as 'deterministic supercritical trees'.

Repeated computer simulations show that (exceedance) size distributions of strings with $n_{s0} < n_{s1}$ are also located between those of the completely bifurcating strings (where $P(\Delta \geq \delta) \propto \delta^{-\epsilon}$ with $\epsilon \approx 1$) and those of strings with $n_{s0} = n_{s1}$ (where $P(\Delta \geq \delta_l)$ is linear), and become close to the former as the B_r increases toward 2 while close to the latter as the B_r decreases toward 1 (Fig. 5). Recall that we consider repetitive strings only with $n_{s0} \leq n_{s1}$. Strings with $n_{s0} > n_{s1}$ show premature growth, i.e., subcritical trees, which are hardly meaningful in tree topology over space. Although there can be other types of deterministic strings, our scope is limited to the repetitive strings discussed so far, which have 'uniform' unit sub-strings.

3.2 Entirely random strings

In these strings, the bifurcation index of each digit is randomly generated without any correlation with those of the other digits. For each bifurcation index, we generate a random number, between 0 and 1, following a uniform distribution. Then we specify a threshold to round off the random number into an integer, either 0 or 1. By adjusting the threshold, we can generate various random bifurcation strings with different B_r. For example, the threshold of 0.5 yields the same number of 0's and 1's on average ($n_0 \approx n_1$), i.e., 'random critical trees'. If the threshold is smaller than 0.5, then there are less 0's than 1's ($n_0 < n_1$) followed by some number of 0's to meet the FSC. The resulting trees are 'random supercritical trees'. Alternately one may use the Bernoulli distribution to generate various classes of random strings.

It had been analytically shown that the load distributions of random critical trees always follow the power law with $\epsilon \approx 0.5$ [23]. For random supercritical trees, De Los Rios [31] claimed that their load distributions follow the power law with $\epsilon \approx 1$. Our repeated (100 times) computer simulations show power law (exceedance) load distributions with $\epsilon = 0.52 \pm 0.05$ for random critical trees, which agrees with [23]. However, for random supercritical trees, fitted power functions to the (exceedance) load distributions exhibit a range of exponent ϵ, which differs from [31] (Fig. 6). The fitted exponent ϵ averaged over 100 simulations exhibits the clear trend related to B_r as $\epsilon \approx 1$ for $B_r = 1.8$, $\epsilon \approx 0.94$ for $B_r = 1.4$, $\epsilon \approx 0.82$ for $B_r = 1.2$, $\epsilon \approx 0.67$ for $B_r = 1.1$, and $\epsilon \approx 0.52$ for $B_r = 1$. This shows that under the restriction of full binary trees, the connectivity structures of random supercritical trees are more diverse than previously thought. Previous studies for random supercritical trees [31] have no constraint of maximum two children per node, which allows denser hierarchical structure, resulting in the exponent ϵ greater than those reported in this study (Fig. 6).

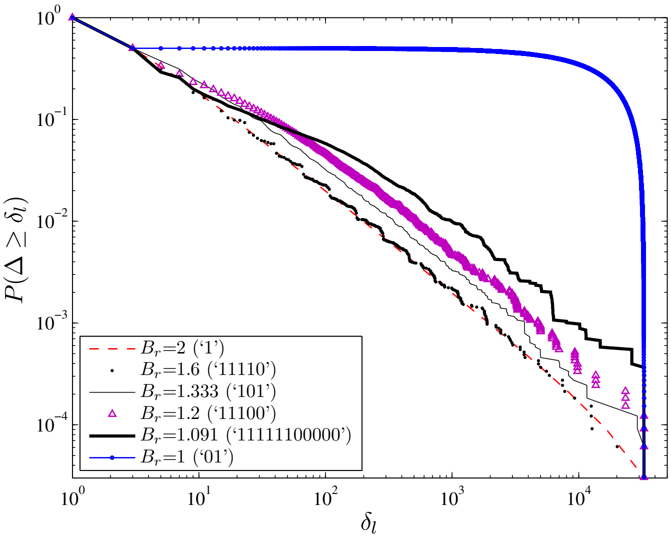

Fig. 5. (Color online) Exceedance size distributions of deterministic trees represented as repetitive strings with various branching ratios B_r. Legend shows unit sub-strings used in this analysis and their corresponding B_r. Exceedance size distributions $P(\Delta \geq \delta_l)$ of deterministic supercritical trees with $1 < B_r < 2$ are located between those of the strict self-similar trees ($B_r = 2$, $P(\Delta \geq \delta_l) \propto \delta_l^{-1}$) and the self-repetitive trees ($B_r = 1$, $P(\Delta \geq \delta_l) = \text{linear}$).

4 Characteristic space-map

In the previous section, we explored tree topology resulting from either deterministic or entirely random bifurcation strings. It is important to notice that observed tree networks do not necessarily belong to these two extreme classes. We may approach this issue from a broader perspective of complex networks. Since randomness is a prevailing property in nature, it is hardly possible for networks to be purely ordered. On the other hand, entirely random networks cannot capture the observed

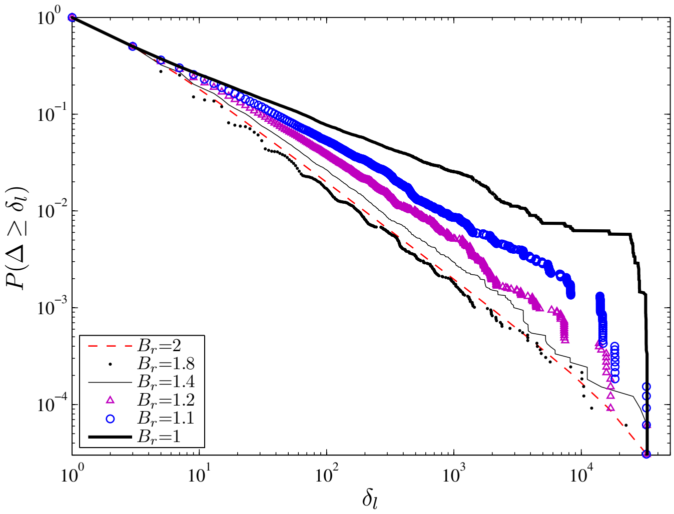

Fig. 6. (Color online) Exceedance size distributions of random trees represented as entirely random strings with various branching ratios B_r. Each distribution is plotted for a randomly selected tree among 100 realizations for a specific B_r. For random critical trees ($B_r = 1$), $\epsilon \approx 0.52$. ϵ increases as B_r increases upto $\epsilon \approx 1$ when $B_r = 2$.

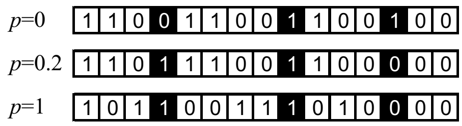

Fig. 7. An illustration of generating a bifurcation string with a specific random-degree p. A deterministic (with unit sub-string '1001') ($p = 0$) and an entirely random ($p = 1$) string of 15 digits are shown. To make a bifurcation string with $p = 0.2$, 3 out of 15 digits ($15 \times 0.2 = 3$) are randomly selected (colored black). The selected 3 bifurcation indexes of the deterministic string are substituted with corresponding 3 bifurcation indexes of the entirely random string. The resulting bifurcation string is based on the deterministic string with unit sub-string '1001' but contains random-degree $p = 0.2$.

clustering of nodes [18]. Therefore, most networks are subject to the organization, somewhere in between these two extremes [18]. If this theory is applicable to full binary trees which form a subset of general complex networks, we postulate that organizations of the observed full binary trees, such as river networks and transformed social networks [11,12], are located between deterministic binary trees, such as the strict self-similar and the self-repetitive trees, and entirely random binary trees. To verify this hypothesis, we explore this intermediate realm.

Similar to an operator used in genetic algorithms [32], we can generate bifurcation strings located in the realm between the deterministic and the entirely random strings by gradually switching bifurcation indexes of deterministic strings with corresponding bifurcation indexes of entirely random strings. The portion p of bifurcation indexes to be switched among the total digit of a deterministic string is called the 'random-degree' of generated bifurcation strings. If random-degree $p = 0$, no bifurcation index is subject to change, resulting in the deterministic string. If $p = 1$, all bifurcation indexes are switched with those of an entirely random string, resulting in the entirely random string.

Binary trees located in the realm between an entirely ordered and an entirely random binary tree can be represented as bifurcation strings with the random-degree between 0 and 1 given to the deterministic string for the entirely ordered binary tree. For example, if $p = 0.4$ is given to a deterministic bifurcation string of 1000 digits, bifurcation indexes of randomly chosen 400 digits are switched with the bifurcation indexes located at the corresponding 400 digits in the entirely random strings, while the other 600 digits keep the same bifurcation indexes as their original values (Fig. 7).

We start from various deterministic strings with different branching ratios B_r and increase the random-degree until p reaches 1 (the entirely random strings). For entirely random strings, we choose only those for random critical trees in this analysis. We analyze trees of fairly consistent size (within $\pm 1\%$ of $M_l = 32767$ and $M_m = 16384$) in a set of simulations. The size adopted here is similar to that of theoretical trees analyzed by De Los Rios [31]. With given size, trees extend at least 14 (only one case of a strict self-similar tree) and up to 16383 (self-repetitive trees with certain unit sub-strings) generations. We also implemented another set of simulations with smaller size of trees (within $\pm 1\%$ of $M_l = 4095$ and $M_m = 2048$), which is similar to the size of observed trees by Guimerà et al. [11] and Arenas et al. [12]. Both sets of simulations gave fundamentally same results and the results from the greater size of trees are presented here. During this experiment, we check how the (exceedance) size distributions vary with p via their fitness to power law and, if they are fitted to the power law, the variation of the exponents (Figs. 8 and 9).

For all deterministic strings used as the basis of our analysis, regardless of the random-degree p, the resulting trees almost always exhibit power law trend in their (exceedance) size distributions (Fig. 8). This is especially interesting for self-repetitive trees, which have linear size distributions (Fig. 8b). Even very little random-degree (e.g., $p = 0.05$) makes $P(\Delta \geq \delta_l)$ of self-repetitive trees converge to the power law. The mechanism that enables this sudden change of network topology can be illustrated by comparing Figures 2 and 4b. If only one bifurcation index (4th digit) of a bifurcation string for Figure 4b is changed, the resulting bifurcation string is 1;10;11;1001;1001;1001;0000. Increased number of 0's at the last generation is to satisfy the FSC. The binary tree generated by this string corresponds to that shown in Figure 2. Note that the difference of only one digit makes the resulting topology very different.

The significance of the above result, i.e., the statistical power law trend is found in most cases regardless of p, is far reaching. This addresses an important notion that, if we take the power law (exceedance) size distribution as

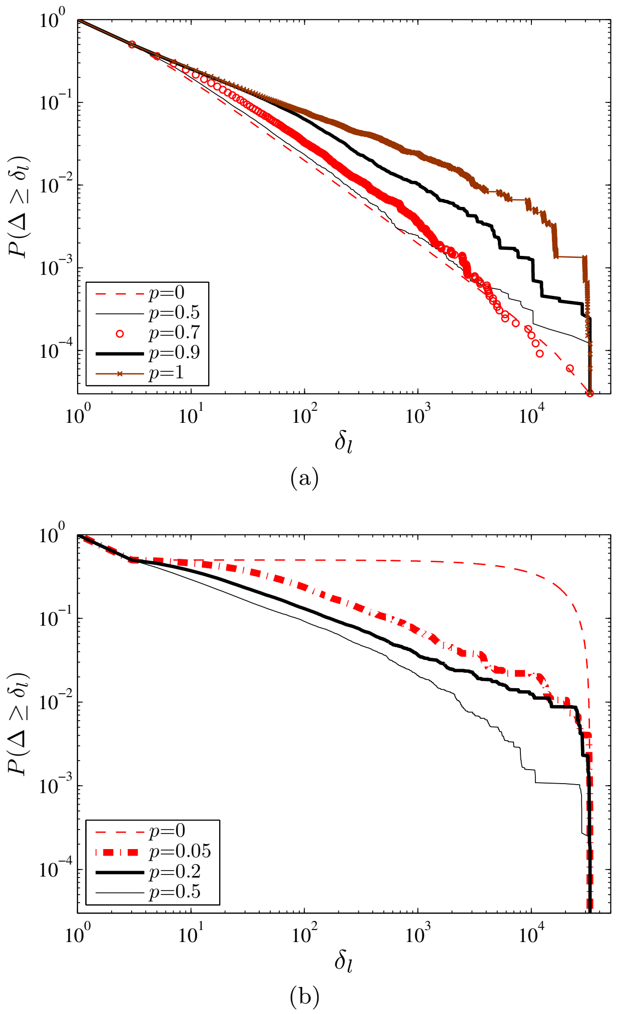

Fig. 8. (Color online) Variation of exceedance size distributions along with random-degree p. Exceedance size distributions of trees based on (a) a strict self-similar tree ($B_r = 2$) and (b) a self-repetitive tree ($B_r = 1$, with unit sub-string '01') are shown. Each distribution is plotted for a randomly selected tree among 30 realizations for each p value. Fitted exponent ϵ, averaged over 30 simulations, is illustrated in Figure 9. (a) As p increases, the $P(\Delta \geq \delta_l)$ distribution keeps the power law trends. The fitted exponent remains as $\epsilon \approx 1$ until p gets close to 1. (b) As p increases, the linear exceedance distribution $P(\Delta \geq \delta_l)$ of the original tree ($p = 0$) rapidly converges to the power law.

an indicator of the statistical self-similarity, the statistical self-similar topology is the inevitable consequence of any full binary tree with few exceptions such as self-repetitive trees. This generalizes the argument based on the analytical derivation discussed in Section 2.

In Figure 8, one may argue how strictly these (exceedance) size distributions follow power law. However, it is worth noting that observed power law (exceedance) size distributions, such as those of river and transformed social networks, show similar behavior of statistical (instead of strict) self-similarity [11,12,14]. They commonly exhibit some deviations for low as well as high δ values (finite size effect). Such deviations are also found even in deterministic trees (Fig. 5) and entirely random trees (Fig. 6).

Once we fit power functions to (exceedance) size distributions, the exponent ϵ shows dependence on p (Fig. 9). Curves which started from various deterministic strings (with unit sub-strings shown in the figure legend) follow clear trends converging around $\epsilon \approx 0.52$ at $p = 1$ (also close to the theoretical value of $\epsilon = 0.5$ [23]). This shows a wide range of ϵ possible for various binary trees. This is contradictory to earlier studies by Arenas et al. [12] who postulated that binary trees belong to two distinct classes based on their size distribution (i.e., one with $\epsilon \approx 0.5$ and the other with $\epsilon \approx 1$) and Caldarelli et al. [17] who argued that any treelike representation should lead to $\epsilon \approx 1$ or rarely 0.5. Note that this study is the first work that investigates the size distribution of trees over the whole range from purely deterministic to entirely random. This results in capturing much more variability in the degree of 'hierarchical density', represented as ϵ, across infinite number of theoretical binary trees than those of the previous studies [17,23,31], which were limited to the class of entirely random trees. Magnitude distributions are very close to those in Figure 9 due to the reason described in Section 2.

At this point, it is worth thinking about the implication of Figure 9. Clear trends of the exponent ϵ in Figure 9 help us locate any binary tree if we know its ϵ. There are several groups of binary trees whose ϵ is known. For river networks, based on the observed $\epsilon = 0.43 \pm 0.03$, Figure 9 indicates that their connectivity structure is far from that of the strict self-similar trees. This is interesting since evident statistical self-similarity of river networks easily tempts us to believe that the connectivity structure of river networks is close to that of strict self-similar trees with inherent randomness. However, as Figure 9 shows, trees generated by increasing random-degree in strict self-similar trees never have ϵ below 0.5. Similarly, some transformed social binary trees as a result of community identification which have $\epsilon \approx 0.5$ also have connectivity structure far from that of strict self-similar trees. However, Figure 9 indicates that the connectivity structure of other transformed binary trees which have $\epsilon \approx 1$, such as scientists networks [12], may be close to that of deterministic supercritical trees.

Figure 9 therefore provides a macroscopic perspective regarding the range of possible binary trees. A specific binary tree, such as a river network and a transformed social network, is merely a point in the space of (p, ϵ). Characteristics other than the hierarchical density (ϵ) can also be used to illustrate such a space. Therefore, these figures may be defined as the characteristic space-map. This allows us to identify the key connectivity structure of binary tree networks.

A noticeable finding on strict self-similar trees is that they keep the original hierarchical density of $\epsilon \approx 1$ even after 70% of their original string is randomized, indicating strong robustness to random noises (Fig. 9). The

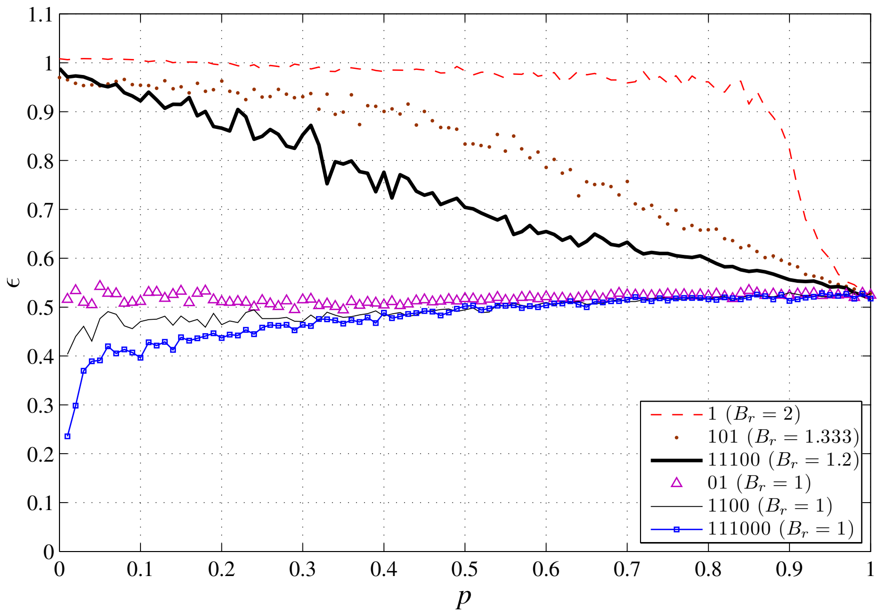

Fig. 9. (Color online) Characteristic space-map showing the variation of fitted exponent ϵ of power law exceedance load distribution according to the random-degree p added in the deterministic strings. Averages over 30 simulations are shown. Legend shows unit sub-strings of deterministic strings ($p = 0$) and their corresponding B_r. Plots are shown only for ranges of high fittness (correlation coefficient > 0.95 without higher order trend) to the power law. As p increases, ϵ of supercritical trees decreases until $\epsilon \approx 0.52$ at $p = 1$. For strict self-similar trees, most variation of size distribution (ϵ) occurs only after $p > 0.8$. In other words, strict self-similar trees keep the original value of $\epsilon \approx 1$ even after 70% of their original string is randomized, indicating more robustness to the disorder (random noise) than other deterministic trees. The decreasing trend of ϵ begins at less p for deterministic supercritical trees of smaller B_r. Contrary to deterministic supercritical trees, ϵ of deterministic critical (self-repetitive) trees slowly increases with p until $\epsilon \approx 0.52$ at $p = 1$. This trend of self-repetitive trees is more obvious for those having bifurcation strings of longer period. Convergence to $\epsilon \approx 0.52$ is faster (occurs at less p) for self-repetitive trees than deterministic supercritical trees.

underlying mechanism that enables such a strong ability in conserving the original signal remains to be seen. This phenomena has potential usefulness in practical engineering applications, which should be another subject of future research. We also found that generally the power law tendency is more evident in exceedance distributions $P(\Delta \geq \delta)$ than $P(\delta)$ over the whole range from purely deterministic to entirely random trees. This is not necessarily consistent with Caldarelli et al. [17] who claimed that if $P(\Delta \geq \delta)$ follows a power law, then $P(\delta)$ also follows a power law. Note that the map in Figure 9 is for trees between deterministic trees and random critical trees. Simply because there are infinite full binary trees, there should be other maps to cover other realms, e.g., the realm between deterministic trees and random supercritical trees.

5 Conclusions

Understanding the topology of binary tree networks is important in that their topology serves as a motif for complex networks and significantly affects flow through the networks. We have devised a technique to represent the tree topology as the bifurcation string and applied this to special binary trees: deterministic supercritical (including strict self-similar) trees, deterministic critical (self-repetitive) trees, random supercritical trees, and random critical trees. Then, we investigate trees located between deterministic and entirely random trees. This is implemented by giving varying random perturbation to basic deterministic trees, using an operator similar to that used in genetic algorithms. This analysis leads to the following conclusions.

The power law (exceedance) size distribution, which is found in almost every full binary tree in various disciplines, is the inevitable result of almost any full binary tree organization. We show this through both a theoretical derivation and numerical simulations using bifurcation strings. If we take the power law (exceedance) size distribution as an indicator of the self-similarity, this leads to an even more significant conclusion that the statistical self-similar topology is an inevitable consequence of any full binary tree organization.

Although the power law itself is inevitable, we show that the fitted exponents (the hierarchical density) vary with clear trends depending on the random-degree p in tree topology. The resulting plot, called the characteristic space-map, for the variation of the hierarchical density as a function of given random-degree provides the macroscopic perspective regarding a range of possible binary trees. The characteristic space-map for the hierarchical density helps explain connectivity structures of observed self-similar trees with very different exponents ϵ. Some social networks with $\epsilon \approx 1$ are close to deterministic supercritical trees with some randomness. On the other hand, an e-mail network [11], a Jazz musician network [12], and river networks, are far from strict self-similar trees. This is surprising since evident self-similarity of these networks easily drives us to assume that their connectivity structure is close to strict self-similar trees with inherent randomness.

Wide and continuous range of exponents (mostly $0.4 \leq \epsilon \leq 1$), found in numerous theoretical full binary trees, indicates the existence of more diverse hierarchical density than previous thought [17,23,31]. We should therefore be cautious of attempting to classify binary trees based on hierarchical density of only a few observed trees. Observed trees, other than full binary trees, still follow power law size distributions [16,33] but without the constraint of full binary tree the fitted exponents can exceed the range of exponents analyzed in this study.

Self-similar topology has been understood as results of evolutionary processes that pursue some form of optimization [7,16,34–41]. However, Paik and Kumar [42] showed that inherent randomness is a sufficient condition for the generation of tree patterns under the evolutionary dynamics. In addition to this finding on evolutionary processes, present study shows that self-similar topology is an inevitable consequence of any full binary tree organization. These findings lead to the conclusion that self-similar patterns in nature are interesting but no specific rules are required for generating such patterns. Explaining such patterns based on certain global rules such as optimality criteria may have limitations.

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Appendix A: Power law size distributions of trees following Horton's laws

In this appendix, we show an analytical derivation of the load distribution for trees that follow Horton's laws. In a tree network, the downstream end junction of an ω order branch is defined as an ω order node. Horton's laws state the average load (the contributing area in river networks) of ω order nodes as:

$$\overline{\delta}_\omega = \overline{\delta}_1 {R_A}^{\omega-1}, \tag{4}$$

and the average number of ω order branches as:

$$\overline{N}_\omega = {R_B}^{\Omega-\omega} \tag{5}$$

where Ω is the order of the root (the outlet in river networks). Constants R_A and R_B are the drainage area ratio and the bifurcation ratio, respectively. Rearranging equation (4) for ω and substituting this into equation (5) yields:

$$\overline{N}_\omega = R_B^{\left(\Omega-1-\frac{\ln\overline{\delta}_\omega-\ln\overline{\delta}_1}{\ln R_A}\right)} \tag{6}$$

which can be written as:

$$\ln\overline{N}_\omega = \left(\Omega - 1 - \frac{\ln\overline{\delta}_\omega - \ln\overline{\delta}_1}{\ln R_A}\right)\ln R_B. \tag{7}$$

Therefore, $\overline{N}_\omega$ follows a power function as:

$$\overline{N}_\omega = [\exp(\Omega-1)]^{\ln R_B}\left(\overline{\delta}_\omega/\overline{\delta}_1\right)^{-\frac{\ln R_B}{\ln R_A}}. \tag{8}$$

Note that normalizing the equation above gives the load distribution as:

$$P\left(\overline{\delta}_\omega\right) = \frac{[\exp(\Omega-1)]^{\ln R_B}}{\overline{\delta}_1^{-\frac{\ln R_B}{\ln R_A}}\sum_{i=1}^{\Omega}\overline{N}_i}\overline{\delta}_\omega^{-\frac{\ln R_B}{\ln R_A}} \tag{9}$$

which is an exact power function with exponent $\epsilon = \frac{\ln R_B}{\ln R_A} - 1$ (comparing with Eq. (1)). The exceedance load distribution is obtained as:

$$P(\Delta \geq \overline{\delta}_\omega) = \frac{[\exp(\Omega-1)]^{\ln R_B}}{-\epsilon\overline{\delta}_1^{-\epsilon-1}\sum_{i=1}^{\Omega}\overline{N}_i}[\delta_\Omega^{-\epsilon} - \overline{\delta}_\omega^{-\epsilon}] \tag{10}$$

which is close to a power function. Therefore, if Horton's laws (Eqs. (4) and (5)) hold for a tree, the (exceedance) size distribution of the tree is subject to follow power law. Similar derivations are also given by Veitzer et al. [43] from a different approach.

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